MAT2384: Ordinary Differential Equations and Numerical Methods

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General Introduction

0.1 Basic Definitions

0.1.1 Differential equation

A differential equation is an equation involving an unknown function and some of its derivatives.

0.1.2 Examples

$$\frac{dy}{dt} = 10y^2 + \sin(t)$$
$$y'' + 10y'y = 0$$

Here y is really $y(t)$. It's easy to spot which are the variables and which are the unknown constants: variables have derivatives.

A derivative is a rate of change, so functions are used because they change (with respect to time, or space or whatever).

0.1.3 Order

The order of the differential equation is the order of the highest derivative appearing in the equation.

0.1.4 Examples

$$\frac{dy}{dt} = 10y^2 + \sin(t)$$

has order 1 while

$$y'' + 10y'y = 0$$

has order 2.

0.1.5 Linear equations

A linear equation is an equation in which the unknown function $y(t)$ and its derivatives appear by themselves. For example,

$$\frac{d^2y}{dt^2} = 10\left(\frac{dy}{dt}\right)^2 + \sin(t)$$

is nonlinear while

$$y'' + 10y' + 2y = 0$$

is linear.

Similarly, the equation

$$y'' + (\sin t)y' + (\cos^2 t)y = 6$$

is linear, because the function y and its derivatives are affected only by addition and scalar (with respect to y) multiplication.

0.1.6 Existence and uniqueness

Existence and uniqueness are important questions. Does a solution for an equation exist? An existence theorem might tell us if it does. Is the solution unique? Some equations may have no solutions at all and some may have infinitely many. Some equations have infinitely many solutions but only one solution that satisfies certain conditions.

0.1.7 Initial-value problem

If we are given a differential equation and all the required conditions that the solution must satisfy at a single point then we have an initial-value problem. For example,

$$y'' + y' + y = 1 \text{ with } y(0) = 1 \text{ and } y'(0) = 3$$

Many functions may satisfy the DE and not the initial conditions. For example, $y(t) = 1$ satisfies the DE and one condition but not the other - so it doesn't satisfy the IVP.

0.1.8 Two-point boundary-value problem

If we are given a differential equation but the conditions are at two different values of t, we have a two-point boundary-value problem. For example,

$$y'' + y' + y = 1 \text{ with } y(0) = 1 \text{ and } y(1) = 3$$

0.1.9 General solutions

If we are not given any conditions then we can only find a general solution that will usually contain one or more unknowns.

Example 1.

$$y'' + y = 0$$

has the solutions

$$y = A\sin(t) + B\cos(t)$$

where A and B are arbitrary constants.

This allows us to add in initial or boundary conditions later.

0.1.10 Initial conditions

Example 2.

$$y'' - y = 0 \text{ with } y(0) = 1, y'(0) = 0.$$

The general solution is

$$y = Ae^x + Be^{-x}$$

Applying initial conditions, we have

$$\begin{aligned} y(0) &= A + B = 1 \\ y' &= Ae^x - Be^{-x} \\ y'(0) &= A - B = 0 \\ \therefore\ A &= B = \frac{1}{2} \end{aligned}$$

Hence the solution to the initial-value problem is

$$y = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

Recall the definitions of cosh and sinh (hyperbolic cosine and hyperbolic sine, respectively):

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$
$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

Note in particular that $\frac{d}{dx}\cosh(x) = \sinh(x)$ and $\frac{d}{dx}\sinh(x) = \cosh(x)$.

These aren't trigonometric functions, but are instead shorthands for exponential functions... but so are $\sin(x)$ and $\cos(x)$ if you include complex numbers.

0.2 Modelling with ODEs

0.2.1 Introduction

An important aspect of the study of ordinary differential equations is the use of such equations in the study of problems in diverse areas such as physical and biological sciences and finance.
The basic idea is that we build a mathematical model of the problem which lets us investigate relevant characteristics while being simple enough to study and analyze.
Anything that changes in some way can be modelled by DEs.
We look at some simple models.

0.3 Growth and decay of populations

Let the population of cod in the North Atlantic at a time t be $C(t)$. Suppose that we assume that the change in population at a time t is proportional to the population itself. Then

$$\frac{dC}{dt} = kC$$

where k is the constant of proportionality. If k is positive then the population will increase if k is negative then the population will decrease.
This is an example of a separable equation. We rearrange the terms to get

$$\frac{dC}{C} = kdt$$

and then integrate both sides to get

$$C(t) = C_0 e^{kt}$$

where $C_0 = C(0)$. This simple model suggests that the cod population will grow (or decrease) exponentially with time. This neglects the fact that a population cannot grow beyond limits set by availability of resources such as food. Let's see how we can incorporate this idea onto a model.
Suppose that we can set a limit to the number of cod based on information about food sources. Let this limit be C_{max}. Then a new model might be

$$\frac{dC}{dt} = kC\left(1 - \frac{C}{C_{max}}\right)$$

We see that this equation is nonlinear. For C small compared to C_{max} we have approximately our original linear equation. As C approaches C_{max} the time derivative goes to zero and so the population levels off at C_{max}.
The solution is of the form

$$C(t) = \frac{C_{max} C_0}{C_0 + (C_{max} - C_0)e^{-kt}}$$

where $C_0 = C(0)$.

A separable equation is one where we can separate the variables, in this case putting all the Cs on one side (including dC) and the t's on the other. The constant k could go on either side. We'll see later how to get this solution. For now, try differentiating and putting it into the original equation with the initial condition to verify that it is the solution.

0.3.1 Compound interest

Example 3. If we invest 100 dollars at 5 percent interest, after one year we have a sum of 100*1.05 and after n years we have a sum of

$$100 \times (1.05)^n$$

In general if we invest an amount D_0 at an interest rate r compounded m times a year for t years we get

$$D_0(1 + r/m)^{mt}$$

How much do we have if we compound continuously?

If we let the number of times that the interest is compounded increase indefinitely, i.e. $m \to \infty$, we get continuous compounding and we see that the capital $D(t)$ after t years is

$$D(t) = \lim_{m\to\infty} D_0(1 + r/m)^{mt}$$

Take the limit:

$$\begin{aligned}
\ln D(t) &= \ln D_0 + \lim_{m\to\infty} mt \ln(1 + r/m) \\
&= \ln D_0 + \lim_{m\to\infty} \frac{\ln(1 + r/m)}{1/(mt)} = \ln D_0 + \frac{0}{0} \\
&= \ln D_0 + \lim_{m\to\infty} \frac{\frac{1}{1+r/m} \cdot \frac{-r}{m^2}}{-\frac{1}{m^2 t}} \qquad \text{(L'Hôpital's Rule)} \\
&= \ln D_0 + \lim_{m\to\infty} \frac{rt}{1 + r/m} \\
&= \ln D_0 + rt.
\end{aligned}$$

Thus

$$D(t) = D_0 e^{rt}$$

This is the same as for our growth and decay example, so the problem of continuously compounded interest can be modelled by the differential equation

$$D' = rD \qquad \text{with} \qquad D(0) = D_0.$$

0.3.2 Radioactive Decay

Some elements decay at a rate which depends on the amount of material present. The more of the element that is present the faster the decay. Thus if $Q(t)$ is the amount of the element present at time t, the rate of decay could be modelled as

$$\frac{dQ}{dt} = -rQ$$

where r is a positive constant that differs from material to material. Thus if $Q(0) = Q_0$ our previous results give us

$$Q(t) = Q_0 e^{-rt}$$

The half-life is the amount of time taken for the sample to decay to half of what it was originally. If T is the half-life then

$$Q(t) = \frac{1}{2} Q_0 = Q_0 e^{-rT}$$

Solve for r in terms of T to get

$$r = -\frac{\log 0.5}{T} = \frac{0.6931}{T}$$

and so we can also express the decay law as

$$Q(t) = Q_0 e^{-0.6931/Tt}$$

6 Laplace Transforms

6.1 Introduction and definition

One way to solve initial-value problems is to transform the problem to a space where things are easier to manage. The Laplace Transform will turn differential equations into linear problems, which are things we know how to solve. The idea is to transform our problem into Laplace space, solve the much easier problem over there, then recover the original solution by means of an inverse.

Definition 6.1. *The Laplace Transform assigns to the function $f(t)$ a new function, $F(s)$ defined by*

$$\boxed{F(s) = \int_0^\infty e^{-st} f(t) dt.}$$

$F(s)$ is called the Laplace transform of $f(t)$ and we write

$$F(s) = \mathcal{L}[f(t)]$$

Remarks:

1. The formula defines a function $F(s)$. That is, it assigns a number $F(s)$ to each numerical value of s. This means that s acts like a constant inside the integral sign.
2. Since the upper limit of integration is infinite, we have an improper integral. Recall from calculus that this is interpreted as a limit of proper integrals:
$$\int_0^\infty e^{-st} f(t) dt = \lim_{h\to\infty} \int_0^h e^{-st} f(t) dt$$
3. Although the integrand may seem complicated at first, the formulas for the transforms of the functions we most often deal with turn out to be quite reasonable. What's more, we only need to derive each of them once.
4. We haven't specified the domain of F. To do so, we'll restrict s if need be in order to ensure the result is finite.

Example 4. Calculate $\mathcal{L}[e^{\lambda t}]$.

From the definition, we have

$$\mathcal{L}[e^{\lambda t}] = \int_0^\infty e^{-st} e^{\lambda t} dt = \lim_{h\to\infty} \int_0^h e^{-(s-\lambda)t} dt.$$

The integral will vary depending on whether $s - \lambda = 0$ or not. If $s = \lambda$, then

$$\mathcal{L}[e^{\lambda t}] = \lim_{h\to\infty} \int_0^h 1 dt = \lim_{h\to\infty} h = \infty$$

If $s \neq \lambda$, then

$$\mathcal{L}[e^{\lambda t}] = \lim_{h\to\infty} \int_0^h e^{-(s-\lambda)t} dt = \lim_{h\to\infty} \left[-\frac{1}{s-\lambda} e^{-(s-\lambda)h} + \frac{1}{s-\lambda} \right]$$

These limits only makes sense if $s > \lambda$. Thus the function $F(s)$ is

$$F(s) = \mathcal{L}[e^{\lambda t}] = \frac{1}{s-\lambda} \qquad \text{for } s > \lambda$$

In particular, if $\lambda = 0$, we have

$$\mathcal{L}[1] = \frac{1}{s} \qquad \text{for } s > 0$$

Example 5. Calculate $\mathcal{L}[\cos \beta t]$ for $\beta \neq 0$.

From the definition, we have

$$\mathcal{L}[\cos \beta t] = \int_0^\infty e^{-st} \cos \beta t dt = \lim_{h \to \infty} \int_0^h e^{-st} \cos \beta t dt.$$

Let $I = \int_0^h e^{-st} \cos \beta t dt$. Then we have

$$u = e^{-st} \qquad\qquad v' = \cos \beta t$$

$$u' = -se^{-st} \qquad\qquad v = \frac{1}{\beta} \sin \beta t$$

$$I = \frac{1}{\beta} e^{-st} \sin \beta t \Big|_0^h + \frac{s}{\beta} \int_0^h e^{-st} \sin \beta t dt$$

$$u = e^{-st} \qquad\qquad v' = \sin \beta t$$

$$u' = -se^{-st} \qquad\qquad v = -\frac{1}{\beta} \cos \beta t$$

$$\begin{aligned}
I &= \frac{1}{\beta} e^{-st} \sin \beta t \Big|_0^h + \frac{s}{\beta} \left[-\frac{1}{\beta} e^{-st} \cos \beta t - \frac{s}{\beta} \int_0^h e^{-st} \cos \beta t dt \right] \\
&= \left[\frac{1}{\beta} e^{-st} \sin \beta t - \frac{s}{\beta^2} e^{-st} \cos \beta t \right]_0^h - \frac{s^2}{\beta^2} I \\
I \left(1 + \frac{s^2}{\beta^2} \right) &= \left[\frac{1}{\beta} e^{-sh} \sin \beta h - \frac{s}{\beta^2} e^{-sh} \cos \beta h \right] - \left[0 - \frac{s}{\beta^2} \right] \\
I &= \left(\frac{\beta^2}{\beta^2 + s^2} \right) \left[\frac{1}{\beta} e^{-sh} \sin \beta h - \frac{s}{\beta^2} e^{-sh} \cos \beta h \right] + \left(\frac{\beta^2}{\beta^2 + s^2} \right) \frac{s}{\beta^2}
\end{aligned}$$

Taking limits as $h \to \infty$, we have

$$\mathcal{L}[\cos \beta t] = \frac{s}{\beta^2 + s^2} \qquad \text{for } s > 0.$$

(We impose the condition $s > 0$ to stop solutions going to infinity.)

Example 6. Find $\mathcal{L}[\sin \beta t]$ for $\beta \neq 0$.

We could do this by mimicking the previous example. But notice that the first integration by parts gave us

$$I = \frac{1}{\beta} e^{-st} \sin \beta t \Big|_0^h + \frac{s}{\beta} \int_0^h e^{-st} \sin \beta t dt$$

The integral on the right is what we want to find in this example. Thus

$$\int_0^h e^{-st} \sin \beta t dt = -\frac{\beta}{s} \left[\frac{1}{\beta} e^{-sh} \sin \beta h \right] + \frac{\beta}{s} I$$

Taking the limit as $h \to \infty$ with $s > 0$, we have

$$\mathcal{L}[\sin \beta t] = \frac{\beta}{s}\mathcal{L}[\cos \beta t] = \frac{\beta}{s}\frac{s}{\beta^2 + s^2} = \frac{\beta}{\beta^2 + s^2} \qquad \text{for } s > 0$$

Example 7. Calculate $\mathcal{L}[t^n]$ where n is a positive integer.

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st}t^n dt = \lim_{h\to\infty}\int_0^h e^{-st}t^n dt$$

Using integration by parts, we have

$$u = t^n \qquad\qquad v' = e^{-st}$$

$$u' = nt^{n-1} \qquad\qquad v = -\frac{1}{s}e^{-st}$$

$$\begin{aligned}\int_0^h e^{-st}t^n dt &= -\frac{t^n}{s}e^{-st}\Big|_0^h + \frac{n}{s}\int_0^h e^{-st}t^{n-1}dt \\ &= -\frac{h^n}{s}e^{-sh} + \frac{n}{s}\int_0^h e^{-st}t^{n-1}dt\end{aligned}$$

Taking the limit as $h \to \infty$, the term outside the integrand will go to zero if $s > 0$ (by L'Hôpital's rule). We thus have

$$\begin{aligned}\lim_{h\to\infty}\int_0^h e^{-st}t^n dt &= -\lim_{h\to\infty}\frac{h^n}{s}e^{-sh} + \lim_{h\to\infty}\frac{n}{s}\int_0^h e^{-st}t^{n-1}dt \\ \mathcal{L}[t^n] &= \frac{n}{s}\mathcal{L}[t^{n-1}] \qquad \text{for } s > 0\end{aligned}$$

We thus have a recursive formula, which can eventually bring us down to $\mathcal{L}[1] = \frac{1}{s}$. That is

$$\begin{aligned}\mathcal{L}[t^n] &= \frac{n}{s}\mathcal{L}[t^{n-1}] = \frac{n(n-1)}{s^2}\mathcal{L}[t^{n-2}] = \cdots = \frac{n!}{s^n}\mathcal{L}[1] \\ &= \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\end{aligned}$$

Example 8. Calculate $\mathcal{L}[3e^{2t} - t^4]$.

By definition, we have

$$\mathcal{L}[3e^{2t} - t^4] = \int_0^\infty e^{-st}(3e^{2t} - t^4)dt$$

We could use integration by parts, but we don't need to, because we know that integrals are linear. Thus

$$\begin{aligned}\int_0^\infty e^{-st}(3e^{2t} - t^4)dt &= 3\int_0^\infty e^{-st}e^{2t}dt - \int_0^\infty e^{-st}t^4 dt \\ &= 3\mathcal{L}[e^{2t}] - \mathcal{L}[t^4] \\ &= 3\frac{1}{s-2} - \frac{4!}{s^5} \\ &= \frac{3}{s-2} - \frac{24}{s^5}\end{aligned}$$

Note that $\mathcal{L}[e^{2t}]$ is defined for $s > 2$ while $\mathcal{L}[t^4]$ is defined for $s > 0$. Thus the new formula will be valid for all values of s that are higher than both 0 and 2; ie for $s > 2$.

This gives us the idea for finding Laplace transforms of linear combinations of any known functions. Because it is an integral, the Laplace transform is linear. That is, for any two functions $f_1(t)$ and $f_2(t)$ and constants c_1 and c_2, we have

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 \mathcal{L}[f_1(t)] + c_2 \mathcal{L}[f_2(t)]$$

Example 9. Find $\mathcal{L}[5 - e^{-7t} + 3\cos 4t]$.

By linearity,

$$\begin{aligned}
\mathcal{L}[5 - e^{-7t} + 3\cos 4t] &= 5\mathcal{L}[1] - \mathcal{L}[e^{-7t}] + 3\mathcal{L}[\cos 4t] \\
&= 5\frac{1}{s} - \frac{1}{s+7} + 3\frac{s}{s^2+16} \\
&= \frac{5}{s} - \frac{1}{s+7} + \frac{3s}{s^2+16} \qquad \text{for } s > 0
\end{aligned}$$

Example 10. Find $\mathcal{L}[\cosh \beta t]$

We have

$$\begin{aligned}
\mathcal{L}[\cosh \beta t] &= \lim_{h\to\infty} \int_0^h e^{-st} \cosh \beta t dt \\
&= \lim_{h\to\infty} \int_0^h e^{-st} \frac{e^{\beta t} + e^{-\beta t}}{2} dt \\
&= \frac{1}{2} \lim_{h\to\infty} \int_0^h \left(e^{(\beta - s)t} + e^{-(\beta+s)t}\right) dt
\end{aligned}$$

If $s = \beta$, the integral is

$$\begin{aligned}
\frac{1}{2} \lim_{h\to\infty} \int_0^h \left(1 + e^{-2\beta t}\right) dt &= \frac{1}{2} \lim_{h\to\infty} h - \frac{e^{-2\beta h}}{2} + \frac{1}{2} \\
&= \infty
\end{aligned}$$

which we don't want, so we can rule this case out. Similarly, if $s = -\beta$, the integral is

$$\begin{aligned}
\frac{1}{2} \lim_{h\to\infty} \int_0^h \left(e^{2\beta t} + 1\right) dt &= \frac{1}{2} \lim_{h\to\infty} \frac{e^{2\beta h}}{2} + h - \frac{1}{2} \\
&= \infty
\end{aligned}$$

so we can also rule this out. If $s \neq \pm\beta$, the integral is

$$\begin{aligned}
\frac{1}{2} \lim_{h\to\infty} \int_0^h \left(e^{(\beta-s)t} + e^{-(\beta+s)t}\right) dt &= \frac{1}{2} \lim_{h\to\infty} \left[\frac{e^{(\beta-s)t}}{\beta - s} - \frac{e^{-(\beta+s)t}}{\beta+s}\right]_0^h \\
&= \frac{1}{2} \lim_{h\to\infty} \left[\left(\frac{e^{(\beta-s)h}}{\beta-s} - \frac{e^{-(\beta+s)h}}{\beta+s}\right) - \left(\frac{1}{\beta-s} - \frac{1}{\beta+s}\right)\right]
\end{aligned}$$

For the integral to converge, we need $\beta - s < 0$ and $\beta + s > 0$. This means that $s > \beta$ and $s > -\beta$. Thus if $s > \beta$, we have

$$\begin{aligned}
\frac{1}{2}\left[0 - 0 - \frac{1}{\beta - s} + \frac{1}{\beta+s}\right] &= \frac{1}{2} \cdot \frac{-(\beta+s) + \beta - s}{(\beta+s)(\beta-s)} \\
&= \frac{1}{2} \cdot \frac{-2s}{\beta^2 - s^2} \\
&= \frac{s}{s^2 - \beta^2}
\end{aligned}$$

Alternate solution:

$$\mathcal{L}[\cosh \beta t] = \frac{1}{2}\left\{\mathcal{L}[e^{\beta t}] + \mathcal{L}[e^{-\beta t}]\right\}$$
$$= \frac{1}{2}\left[\frac{1}{s-\beta} + \frac{1}{s+\beta}\right]$$

The domain of definition is the intersection of $\{s > \beta\}$ and $\{s > -\beta\}$. Hence this is defined for $s > \beta$.

Exercise. Show that

$$\mathcal{L}[\sinh \beta t] = \frac{\beta}{s^2 - \beta^2}$$

The Laplace transform turns out to be very useful for functions that may be defined in pieces.

Example 11. Find $L[u_a(t)]$ where $a > 0$ and

$$u_a(t) = \begin{cases} 0 & \text{if } 0 \le t < a \\ 1 & \text{if } a \le t. \end{cases}$$

By definition

$$\begin{aligned}
\mathcal{L}[u_a(t)] &= \int_0^\infty e^{-st} u_a(t)dt \\
&= \int_0^a e^{-st} u_a(t)dt + \int_a^\infty e^{-st} u_a(t)dt \\
&= \int_0^a e^{-st} 0dt + \int_a^\infty e^{-st} 1dt \\
&= \lim_{h\to\infty} \int_a^h e^{-st} dt \\
&= \lim_{h\to\infty} \begin{cases} h-a & \text{if } s = 0 \\ -\frac{e^{-sh}}{s} + \frac{e^{-sa}}{s} & \text{if } s \ne 0 \end{cases} \\
&= \frac{e^{-sa}}{s} \qquad \text{for } s > 0
\end{aligned}$$

The final step in being able to apply the Laplace transform to initial-value problems will require us to retrieve a function from its Laplace transform.

Definition 6.2. *If $F(s) = \mathcal{L}[f(t)]$, then we say that $f(t)$ is an inverse Laplace transform of $F(s)$ and write*

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

The relationship between $\mathcal{L}$ and $\mathcal{L}^{-1}$ is like the relationship between differentiation and integration. Just as a table of integrals starts from a "backward" reading of differentiation formulas, so an inverse transform table begins with transform formulas read backwards.

We can thus summarise our results thus far.

Laplace transform	Inverse transform
$\mathcal{L}\left[e^{\lambda t}\right] = \dfrac{1}{s-\lambda}$	$\mathcal{L}^{-1}\left[\dfrac{1}{s-\lambda}\right] = e^{\lambda t}$
$\mathcal{L}[1] = \dfrac{1}{s}$	$\mathcal{L}^{-1}\left[\dfrac{1}{s}\right] = 1$
$\mathcal{L}[\cos\beta t] = \dfrac{s}{s^2+\beta^2}$	$\mathcal{L}^{-1}\left[\dfrac{s}{s^2+\beta^2}\right] = \cos\beta t$
$\mathcal{L}[\sin\beta t] = \dfrac{\beta}{s^2+\beta^2}$	$\mathcal{L}^{-1}\left[\dfrac{\beta}{s^2+\beta^2}\right] = \sin\beta t$
$\mathcal{L}\left[\dfrac{t^{n-1}}{(n-1)!}\right] = \dfrac{1}{s^n}$	$\mathcal{L}^{-1}\left[\dfrac{1}{s^n}\right] = \dfrac{t^{n-1}}{(n-1)!}$
$\mathcal{L}[\cosh\beta t] = \dfrac{s}{s^2-\beta^2}$	$\mathcal{L}^{-1}\left[\dfrac{s}{s^2-\beta^2}\right] = \cosh\beta t$
$\mathcal{L}[\sinh\beta t] = \dfrac{\beta}{s^2-\beta^2}$	$\mathcal{L}^{-1}\left[\dfrac{\beta}{s^2-\beta^2}\right] = \sinh\beta t$

Recall that $\mathcal{L}$ is linear. Thus if $\mathcal{L}[f_1(t)] = F_1(s)$ and $\mathcal{L}[f_2(t)] = F_2(s)$, then $\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$. Read backwards, this gives

$$\begin{aligned}\mathcal{L}^{-1}[c_1F_1(s) + c_2F_2(s)] &= c_1 f_1(t) + c_2 f_2(t)\\ &= c_1\mathcal{L}^{-1}[F_1(s)] + c_2\mathcal{L}^{-1}[F_2(s)]\end{aligned}$$

Thus $\mathcal{L}^{-1}$ is linear. Hence if we know the inverse transforms of some basic functions, we can find inverse transforms of their linear combinations.

Example 12. Find $\mathcal{L}^{-1}\left[\frac{2}{s+3} - \frac{6}{s^2+25} + \frac{1}{s^7}\right]$.

Using linearity, we have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2}{s+3} - \frac{6}{s^2+25} + \frac{1}{s^7}\right] &= \mathcal{L}^{-1}\left[\frac{2}{s+3}\right] - \mathcal{L}^{-1}\left[\frac{6}{s^2+25}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^7}\right]\\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - \frac{6}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right] + \frac{1}{6!}\mathcal{L}^{-1}\left[\frac{6!}{s^7}\right]\\ &= 2e^{-3t} - \frac{6}{5}\sin 5t + \frac{1}{720}t^6\end{aligned}$$

6.2 Initial-value problems

We want to solve an ODE with constant coefficients

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = g(t).$$

The idea is for the Laplace transform to change a differential equation into an algebraic equation when the transform is applied to both sides. Since $\mathcal{L}$ is linear, we have

$$a_n\mathcal{L}[x^{(n)}] + a_{n-1}\mathcal{L}[x^{(n-1)}] + \cdots + a_1\mathcal{L}[x'] + a_0\mathcal{L}[x] = \mathcal{L}[g(t)].$$

We know how to find $\mathcal{L}[g(t)]$ for a number of choices of $g(t)$. However, we need a way to deal with the terms $\mathcal{L}[x^{(k)}]$.

Let's start by considering $\mathcal{L}[x'(t)]$. By definition,

$$\mathcal{L}[x'(t)] = \int_0^\infty e^{-st}x'(t)dt = \lim_{h\to\infty}\int_0^h e^{-st}x'(t)dt$$

We use integration by parts:

$$u = e^{-st} \qquad\qquad v' = x'(t)$$
$$u' = -se^{-st} \qquad\qquad v = x(t)$$

$$\begin{aligned}
\mathcal{L}[x'(t)] &= \lim_{h\to\infty} \left(e^{-st}x(t)\Big|_0^h + s\int_0^h e^{-st}x(t)dt \right) \\
&= \lim_{h\to\infty} e^{-sh}x(h) - x(0) + s\mathcal{L}[x(t)]
\end{aligned}$$

For the functions we are interested in,

$$\lim_{h\to\infty} e^{-sh}x(h) = 0$$

as long as s is sufficiently large. We thus have the following:

$\boxed{\textbf{First Differentiation Formula (k=1): } \mathcal{L}[x'(t)] = s\mathcal{L}[x] - x(0).}$

Example 13. Solve the initial-value problem $x' = t$, $x(0) = 2$ using a Laplace transform.

Applying $\mathcal{L}$ to both sides of the IDE, we have

$$\begin{aligned}
\mathcal{L}[x'(t)] &= \mathcal{L}[t] \\
s\mathcal{L}[x] - x(0) &= \frac{1}{s^2} \\
s\mathcal{L}[x] - 2 &= \frac{1}{s^2} && \text{using the initial condition} \\
\mathcal{L}[x] &= \frac{1}{s^3} + \frac{2}{s} \\
x &= \mathcal{L}^{-1}\left[\frac{1}{s^3} + \frac{2}{s}\right] && \text{taking the inverse transform} \\
&= \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] && \text{since } \mathcal{L}^{-1} \text{ is linear} \\
&= \frac{1}{2}t^2 + 2 && \text{from the table}
\end{aligned}$$

How to deal with higher order derivatives?
For $k = 2$, we note that $x''(t) = (x'(t))'$. Thus

$$\begin{aligned}
\mathcal{L}[x''(t)] &= \mathcal{L}[(x'(t))'] \\
&= s\mathcal{L}[x'(t)] - x'(0) && \text{applying the first differentiation formula} \\
&= s\Big(s\mathcal{L}[x(t)] - x(0)\Big) - x'(0) \\
&= s^2\mathcal{L}[x] - sx(0) - x'(0)
\end{aligned}$$

Repeated application of this process yields the following general formula.

$\boxed{\textbf{First Differentiation Formula: } \mathcal{L}[x^{(k)}] = s^k\mathcal{L}[x] - s^{k-1}x(0) - s^{k-2}x'(0) - \cdots - sx^{(k-2)}(0) - x^{(k-1)}(0)}$

Example 14. Solve the initial-value problem

$$x''' - x'' = 0, \qquad\qquad x(0) = x'(0) = x''(0) = 3$$

Applying $\mathcal{L}$ to both sides of the ODE, we get

$$\mathcal{L}[x'''] - \mathcal{L}[x''] = \mathcal{L}[0] = 0$$

By the differentiation formula,

$$\begin{aligned}
\mathcal{L}[x'''] &= s^3\mathcal{L}[x] - s^2x(0) - sx'(0) - x''(0) \\
&= s^3\mathcal{L}[x] - 3s^2 - 3s - 3 \\
\mathcal{L}[x''] &= s^2\mathcal{L}[x] - sx(0) - x'(0) \\
&= s^2\mathcal{L}[x] - 3s - 3
\end{aligned}$$

We thus have

$$\begin{aligned}
\mathcal{L}[x'''] - \mathcal{L}[x''] &= 0 \\
\Big(s^3\mathcal{L}[x] - 3s^2 - 3s - 3\Big) - \Big(s^2\mathcal{L}[x] - 3s - 3\Big) &= 0 \\
(s^3 - s^2)\mathcal{L}[x] - 3s^2 &= 0 \\
\mathcal{L}[x] &= \frac{3s^2}{s^3 - s^2} \\
&= \frac{3}{s-1}
\end{aligned}$$

Taking the inverse transform, we have

$$\begin{aligned}
x &= \mathcal{L}^{-1}\left[\frac{3}{s-1}\right] \\
&= 3\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] \\
&= 3e^t
\end{aligned}$$

This example illustrates the three steps in solving an initial-value problem by Laplace transforms:

1. Transform the ODE, incorporating initial data, by means of the first differentiation formula
2. Solve algebraically for $\mathcal{L}[x]$ in terms of s
3. Obtain x as the inverse transform of $\mathcal{L}[x]$

In these examples, the third step was unusually easy, because we recognised the inverse transform from the table. In most cases, we need to rewrite $\mathcal{L}[x]$ in order to recognise its inverse transform. This is done by means of partial fraction decomposition of quotients of polynomials.

6.2.1 Partial Fraction Decomposition

Each polynomial $q(s)$ with real coefficients can, at least in theory, be factored as a number (the leading coefficient) times a product of irreducible polynomials of two kinds: **linear factors** $s - a$, where a is a real root of the polynomial; and **irredicible quadratic factors** $s^2 + bs + c$ (with $b^2 < 4c$), corresponding to pairs of complex roots. If $p(s)$ is a polynomial whose degree is strictly less than the degree of $q(s)$, then the rational expression $p(s)/q(s)$ can be written as a sum according to the following rules:

i. If $(s-a)^m$ is the highest power of $s - a$ that divides $q(s)$, then the sum should include terms of the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}$$

ii. If $(s^2+bs+c)^m$ is the highest power of the irreducible quadratic s^2+bs+c that divides $q(s)$, then the sum should include terms of the form

$$\frac{B_1 s + C_1}{s^2+bs+c} + \frac{B_2 s + C_2}{(s^2+bs+c)^2} + \cdots + \frac{B_m s + C_m}{(s^2+bs+c)^m}$$

In general, we obtain the partial fraction decomposition of a rational expression $p(s)/q(s)$, once we know how to factor $q(s)$, by first using long division to rewrite the original quotient as a polynomial plus a new quotient whose numerator has degree less than the denominator $q(s)$. We then write this new quotient as a sum of terms of the form (i) and (ii), corresponding to all the factors of $q(s)$.

Example 15. Solve the initial-value problem

$$x' - x = 2\sin t, \qquad x(0) = 0$$

First transform both sides of the ODE.

$$\begin{aligned} \mathcal{L}[x'] - \mathcal{L}[x] &= \mathcal{L}[2\sin t] \\ (s\mathcal{L}[x] - x(0)) - \mathcal{L}[x] &= \frac{2}{s^2+1} \\ (s-1)\mathcal{L}[x] &= \frac{2}{s^2+1} \end{aligned}$$

Next, solve for $\mathcal{L}[x]$.

$$\begin{aligned} \mathcal{L}[x] &= \frac{2}{(s-1)(s^2+1)} \\ x &= \mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+1)}\right] \end{aligned}$$

To find the inverse transform, we look for a partial fraction decomposition of $\mathcal{L}[x]$.
This is of the form

$$\begin{aligned} \frac{2}{(s-1)(s^2+1)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \\ 2 &= A(s^2+1) + (Bs+C)(s-1) \end{aligned}$$

Since A, B and C are constants, they must hold true for any values of s. We can thus choose s judiciously.

$$\begin{aligned} s = 1: \qquad & 2 = A(2) + 0 \\ & A = 1 \\ s = 0: \qquad & 2 = A + (0 + C)(-1) \\ & 2 = 1 - C \\ & C = -1 \end{aligned}$$

The choices $s = 1$ and $s = 0$ are "smart" choices, because they eliminate unknown constants. The final choice of s has no obvious "smart" choice, but any s will do, since we know the other constants. Thus

$$\begin{aligned} s = -1: \qquad & 2 = A(2) + (-B + C)(-2) \\ & 2 = 2 + 2B + 2 \\ & B = -1 \end{aligned}$$

Thus the partial fraction decomposition is

$$\begin{aligned}\frac{2}{(s-1)(s^2+1)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \\ &= \frac{1}{s-1} + \frac{-s-1}{s^2+1}\end{aligned}$$

We can thus take the inverse transform in order to find the solution.

$$\begin{aligned}x &= \mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+1)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-1} + \frac{-s-1}{s^2+1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\ &= e^t - \cos t - \sin t\end{aligned}$$

Example 16. Solve the intial value problem

$$x'' - x = 0, \qquad\qquad x(0) = 3, x'(0) = 1$$

First transform both sides of the ODE.

$$\begin{aligned}\mathcal{L}[x''] - \mathcal{L}[x] &= 0 \\ \Big(s^2\mathcal{L}[x] - sx(0) - x'(0)\Big) - \mathcal{L}[x] &= 0 \\ \Big(s^2\mathcal{L}[x] - 3s - 1\Big) - \mathcal{L}[x] &= 0 \\ (s^2-1)\mathcal{L}[x] &= 3s+1 \\ \mathcal{L}[x] &= \frac{3s+1}{s^2-1}\end{aligned}$$

Next, we find the partial fraction decomposition of $\mathcal{L}[x]$.

$$\begin{aligned}\frac{3s+1}{s^2-1} &= \frac{3s+1}{(s+1)(s-1)} \\ &= \frac{A}{s+1} + \frac{B}{s-1} \\ 3s+1 &= A(s-1) + B(s+1)\end{aligned}$$

Thus

$$\begin{aligned}s = 1: &\qquad 4 = B(2) \\ &\qquad B = 2 \\ s = -1: &\qquad -2 = A(-2) \\ &\qquad A = 1\end{aligned}$$

We can thus use the inverse transform to solve for x.

$$\begin{aligned} x &= \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{2}{s-1}\right] \\ &= e^{-t} + 2e^{t} \end{aligned}$$

Example 17. Solve the initial-value problem

$$x'' - 2x' = 4, \qquad x(0) = -1, x'(0) = 2$$

$$\begin{aligned} \mathcal{L}[x''] - 2\mathcal{L}[x'] &= \mathcal{L}[4] \\ \left(s^2\mathcal{L}[x] - sx(0) - x'(0)\right) - 2\left(s\mathcal{L}[x] - x(0)\right) &= \frac{4}{s} \\ \left(s^2\mathcal{L}[x] + s - 2\right) - 2\left(s\mathcal{L}[x] + 1\right) &= \frac{4}{s} \\ (s^2 - 2s)\mathcal{L}[x] &= \frac{4}{s} + 4 - s \\ (s^2 - 2s)\mathcal{L}[x] &= \frac{4 + 4s - s^2}{s} \\ \mathcal{L}[x] &= \frac{4 + 4s - s^2}{s^2(s-2)} \end{aligned}$$

Using partial fractions, we have

$$\begin{aligned} \frac{4 + 4s - s^2}{s^2(s-2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} \\ 4 + 4s - s^2 &= As(s-2) + B(s-2) + Cs^2 \end{aligned}$$

Hence

$$\begin{aligned} s = 0: \qquad & 4 = 0 + B(-2) + 0 \\ & B = -2 \\ s = 2: \qquad & 8 = 0 + 0 + 4C \\ & C = 2 \\ s = 1 \qquad & 7 = A(-1) + B(-1) + C \\ & 7 = -A + 2 + 2 \\ & A = -3 \end{aligned}$$

The solution is thus

$$\begin{aligned} x &= \mathcal{L}^{-1}\left[\frac{4 + 4s - s^2}{s^2(s-2)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{-3}{s} + \frac{-2}{s^2} + \frac{2}{s-2}\right] \\ &= -3\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] \\ &= -3 - 2t + 2e^{2t} \end{aligned}$$

6.2.2 Laplace transforms of integrals

Note that the function

$$g(t) = \int_0^t f(t)dt$$

satisfies $g'(t) = f(t)$ and $g(0) = 0$. Taking Laplace transforms, we get

$$\begin{aligned} s\mathcal{L}[g(t)] - g(0) &= \mathcal{L}[f(t)] \\ \mathcal{L}[g(t)] &= \frac{1}{s}\mathcal{L}[f(t)] \end{aligned}$$

Setting $F(s) = \mathcal{L}[f(t)]$, we see that

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

Alternately,

$$\mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t \mathcal{L}^{-1}[F(s)]dt$$

Example 18. Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+\beta^2)}\right]$.

We have

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s(s^2+\beta^2)}\right] &= \int_0^t \mathcal{L}^{-1}\left[\frac{1}{s^2+\beta^2}\right] dt \\ &= \frac{1}{\beta}\int_0^t \sin\beta t dt \\ &= -\frac{1}{\beta^2}\cos\beta t + \frac{1}{\beta^2} \end{aligned}$$

Example 19. Find $\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right]$.

We can deal with the $\frac{1}{s^2}$ term by integrating twice:

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right] &= \int_0^t\int_0^t \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] dtdt \\ &= \int_0^t\int_0^t e^{2t}dtdt \\ &= \int_0^t \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) dt \\ &= \left[\frac{e^{2t}}{4} - \frac{t}{2}\right]_0^t \\ &= \left[\left(\frac{e^{2t}}{4} - \frac{t}{2}\right) - \left(\frac{1}{4} - 0\right)\right] \\ &= \frac{e^{2t}}{4} - \frac{t}{2} - \frac{1}{4} \end{aligned}$$

6.3 First shift formula

6.3.1 t-shifting

Suppose we know $F(s) = \mathcal{L}[f(t)]$. Then the Laplace transform of $e^{\alpha t} f(t)$ is

$$\begin{aligned} \mathcal{L}[e^{\alpha t} f(t)] &= \int_0^\infty e^{-st} e^{\alpha t} f(t) dt \\ &= \int_0^\infty e^{-(s-\alpha)t} f(t) dt \\ &= F(s-\alpha) \end{aligned}$$

That is, multiplying a function by an exponential is equivalent to a translation in the Laplace transform.

First Shift Formula (t-shifting): If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{\alpha t} f(t)] = F(s-\alpha)$

Example 20. Find $\mathcal{L}[t^3 e^{2t}]$.

We know that

$$\mathcal{L}[t^3] = F(s) = \frac{6}{s^4}.$$

Thus, by the first shift formula, $\mathcal{L}[t^3 e^{2t}]$ is obtained from $\mathcal{L}[t^3]$ by replacing s with $s-2$. Hence

$$\mathcal{L}[t^3 e^{2t}] = F(s-2) = \frac{6}{(s-2)^4}$$

Example 21. Find $\mathcal{L}[e^{-t} \cos 3t]$.

We know

$$\mathcal{L}[\cos 3t] = F(s) = \frac{s}{s^2+9}$$

Thus, replacing s with $s+1$, we have

$$\mathcal{L}[e^{-t} \cos 3t] = F(s+1) = \frac{s+1}{(s+1)^2+9}$$

Of course, every transform can be rewritten as an inverse transform formula. We can thus rewrite the first shift formula as

$$\mathcal{L}^{-1}[F(s-\alpha)] = e^{\alpha t} \mathcal{L}^{-1}[F(s)].$$

If we replace s with $s+\alpha$, then we obtain a formula that is easier to work with.

First shift formula (inverse version): $\mathcal{L}^{-1}[F(s)] = e^{\alpha t} \mathcal{L}^{-1}[F(s+\alpha)]$

Example 22. Find $\mathcal{L}^{-1}\left[\frac{3}{(s-2)^5}\right]$

We know how to deal with inverse transform powers of s, so substituting $s+2$ for s would turn our problem into one of this type. Thus

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3}{(s-2)^5}\right] &= e^{2t} \mathcal{L}^{-1}\left[\frac{3}{(s+2-2)^5}\right] \\ &= e^{2t} \mathcal{L}^{-1}\left[\frac{3}{s^5}\right] \\ &= \frac{3}{4!} e^{2t} t^4 \\ &= \frac{1}{8} e^{2t} t^4 \end{aligned}$$

Example 23. Find $\mathcal{L}^{-1}\left[\dfrac{s}{(s-1)^2+4}\right]$.

If the denominator had the form s^2+4, we could handle this using trigonometric functions. Therefore we try the substitution of $s+1$ for s, which changes $(s-1)^2$ into s^2.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s-1)^2+4}\right] &= e^t\mathcal{L}^{-1}\left[\frac{s+1}{(s+1-1)^2+4}\right]\\
&= e^t\mathcal{L}^{-1}\left[\frac{s}{s^2+4}+\frac{1}{s^2+4}\right]\\
&= e^t\cos 2t+\frac{1}{2}e^t\sin 2t
\end{aligned}$$

In general, to deal with terms of the form $\dfrac{Bs+C}{s^2+bs+c}$, we first check to see whether the denominator can be factored. If it can, we use partial fractions; if it can't, we complete the square and shift.

Example 24. Solve the initial-value problem

$$x''+2x'+2x=25te^t, \qquad x(0)=x'(0)=0$$

Transforming both sides of the ODE, we have

$$\begin{aligned}
s^2\mathcal{L}[x]+2s\mathcal{L}[x]+2\mathcal{L}[x] &= \frac{25}{(s-1)^2}\\
\mathcal{L}[x] &= \frac{25}{(s-1)^2(s^2+2s+2)}\\
&= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{Cs+D}{s^2+2s+2}
\end{aligned}$$

Thus using partial fractions, we have

$$\begin{aligned}
& & 25 &= A(s-1)(s^2+2s+2)+B(s^2+2s+2)+(Cs+D)(s-1)^2\\
&s=1 & 25 &= 5B\\
& & B &= 5\\
&s=0 & 25 &= A(-1)(2)+5(2)+D(1)\\
& & D &= 2A+15\\
&s=-1 & 25 &= A(-2)(1)+5(1)+(-C+D)(4)\\
& & 20 &= -2A-4C+4D\\
& & &= -2A-4C+4(2A+15)\\
& & -40 &= 6A-4C\\
& & C &= \frac{3}{2}A+10
\end{aligned}$$

Finally,

$$s = 2 \qquad \begin{aligned} 25 &= A(1)(10) + 5(10) + (2C + D)(1) \\ -25 &= 10A + 2C + D \\ -25 &= 10A + 2\left(\frac{3}{2}A + 10\right) + (2A + 15) \\ -25 &= 10A + 3A + 20 + 2A + 15 \\ -60 &= 15A \\ A &= -4 \\ C &= \frac{3}{2}(-4) + 10 = 4 \\ D &= 2(-4) + 15 = 7 \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}[x] &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs + D}{s^2 + 2s + 2} \\ &= \frac{-4}{s-1} + \frac{5}{(s-1)^2} + \frac{4s + 7}{s^2 + 2s + 2} \\ x &= \mathcal{L}^{-1}\left[\frac{-4}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{5}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] \\ &= -4\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 5\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] \\ &= -4e^t + 5te^t + \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] \end{aligned}$$

using t-shifting for the first two terms. The third term requires us to complete the square in the denominator.

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{4s+7}{s^2+2s+2}\right] &= \mathcal{L}^{-1}\left[\frac{4s+7}{(s+1)^2+1}\right] \\ &= e^{-t}\mathcal{L}^{-1}\left[\frac{4(s-1)+7}{s^2+1}\right] && \text{by } t\text{-shifting} \\ &= e^{-t}\mathcal{L}^{-1}\left[\frac{4s+3}{s^2+1}\right] \\ &= 4e^{-t}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 3e^{-t}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\ &= 4e^{-t}\cos t + 3e^{-t}\sin t \end{aligned}$$

Thus the solution of the initial-value problem is

$$x = -4e^t + 5te^t + 4e^{-t}\cos t + 3e^{-t}\sin t$$

6.3.2 Second differentiation formula

Another important formula comes from differentiating the Laplace transform. By definition,

$$\frac{d}{ds}\mathcal{L}[f(t)] = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

For functions we are interested in (piecewise continuous and of exponential order), the differentiation can be carried out inside the integral sign. Thus

$$\begin{aligned}\frac{d}{ds}\mathcal{L}[f(t)] &= \int_0^\infty \left[\frac{\partial}{\partial s}e^{-st}f(t)\right]dt\\ &= -\int_0^\infty e^{-st}tf(t)dt\\ &= -\mathcal{L}[tf(t)]\\ \text{Conversely, } \mathcal{L}[tf(t)] &= -\frac{d}{ds}\mathcal{L}[f(t)]\end{aligned}$$

Repeated application of this formula gives a more general version.

$$\boxed{\textbf{Second differentiation formula: } \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)]}$$

Example 25. Find $\mathcal{L}[te^{2t}\cos 3t]$

First we use the second differentiation formula to find $\mathcal{L}[t\cos 3t]$:

$$\begin{aligned}\mathcal{L}[t\cos 3t] &= -\frac{d}{ds}\mathcal{L}[\cos 3t]\\ &= -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)\\ &= -\frac{s^2+9-s(2s)}{(s^2+9)^2}\\ &= \frac{s^2-9}{(s^2+9)^2}\end{aligned}$$

We now use the first shift formula to get

$$\begin{aligned}\mathcal{L}[te^{2t}\cos 3t] &= \frac{(s-2)^2-9}{((s-2)^2+9)^2}\\ &= \frac{s^2-4s-5}{(s^2-4s+13)^2}\end{aligned}$$

6.3.3 Integral of the transform

Let $F(s) = \mathcal{L}[f(t)]$. Then

$$\begin{aligned}\int_s^\infty F(x)dx &= \int_s^\infty \int_0^\infty e^{-sx}f(x)dxds\\ &= \int_0^\infty \int_s^\infty e^{-sx}f(x)dsdx\\ &= \int_0^\infty \lim_{h\to\infty}\int_s^h e^{-sx}f(x)dsdx\\ &= \int_0^\infty \lim_{h\to\infty}\left[\frac{e^{-sx}f(x)}{-x}\right]_s^h dx\\ &= \int_0^\infty \lim_{h\to\infty}\left[\frac{e^{-hx}f(x)}{-x}+\frac{e^{-sx}f(x)}{x}\right]dx\\ &= \int_0^\infty \frac{e^{-sx}f(x)}{x}dx\\ &= \mathcal{L}\left[\frac{f(t)}{t}\right]\end{aligned}$$

Example 26. Find $\mathcal{L}\left[\dfrac{\sinh 3t}{t}\right]$.

Note that the Laplace transform of $\sinh 3t$ is only valid for $s > 3$. We have

$$\mathcal{L}\left[\frac{\sinh 3t}{t}\right] = \int_s^\infty \frac{3}{x^2-9}dx$$

Using partial fractions, we have

$$\begin{aligned}
& & \frac{3}{(x+3)(x-3)} &= \frac{A}{x+3} + \frac{B}{x-3} & \\
& & 3 &= A(x-3) + B(x+3) & \\
x = 3: & & 3 &= B(6) & B = \frac{1}{2} \\
x = -3: & & 3 &= A(-6) & A = -\frac{1}{2} \\
& & \frac{3}{(x+3)(x-3)} &= \frac{-1/2}{x+3} + \frac{1/2}{x-3} &
\end{aligned}$$

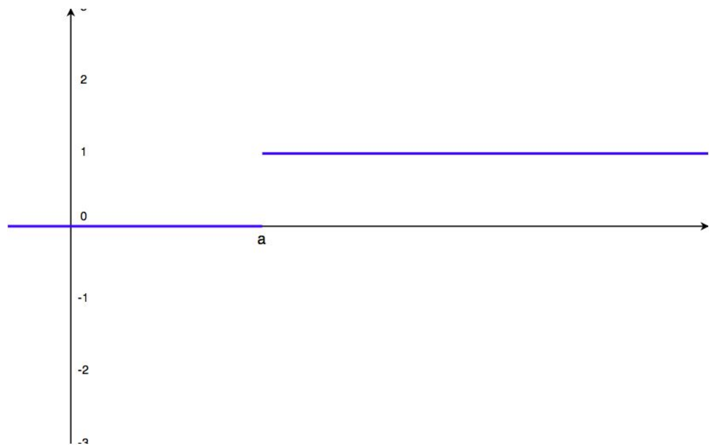
Thus

$$\begin{aligned}
\mathcal{L}\left[\frac{\sinh 3t}{t}\right] &= \lim_{h\to\infty} \frac{1}{2}\int_s^h -\frac{1}{x+3} + \frac{1}{x-3}dx \\
&= \lim_{h\to\infty} \frac{1}{2}\Big[-\ln(x+3) + \ln(x-3)\Big]_s^h \qquad \text{(this is well-defined since } x > 3) \\
&= \lim_{h\to\infty} \frac{1}{2}\left[-\ln\frac{x+3}{x-3}\right]_s^h \\
&= \lim_{h\to\infty} \frac{1}{2}\left[-\ln\frac{h+3}{h-3} + \ln\frac{s+3}{s-3}\right] \\
&= \frac{1}{2}\ln\frac{s+3}{s-3}
\end{aligned}$$

6.4 Piecewise continuous functions

Many phenomena are best defined by functions in pieces: circuits with switches, doses of drugs, sudden shocks. Our first task is to develop a better notation for functions defined in pieces. This is accomplished through the use of the unit step function:

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$



We already found that $\mathcal{L}[u_a(t)] = \frac{e^{-sa}}{s}$.

The unit step function has the effect of a mathematical "on" switch at $t = a$. If we multiply a function $f(t)$ by $u_a(t)$, then product will be zero until $t = a$ and will switch to $f(t)$ thereafter:

$$u_a(t)f(t) = \begin{cases} 0 & \text{if } t < a \\ f(t) & \text{if } t \geq a \end{cases}$$

Thus, for example, the function

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ e^{-t} & \text{if } t \geq 2 \end{cases}$$

can be rewritten as

$$g(t) = u_2(t)e^{-t}$$

Of course, we are often interested in functions that switch from one nonzero formula to another, such as

$$g(t) = \begin{cases} t^2 & \text{if } t < 3 \\ e^{-t} & \text{if } t \geq 3 \end{cases}$$

We can think of this as a function that starts out equal to t^2. At time $t = 3$, a switch does two things: it turns off the first formula and turns on the second. To turn off the formula $g(t) = t^2$ at $t = 3$, we subtract $u_3(t)t^2$. To turn on the formula $g(t) = e^{-t}$ at $t = 3$, we add $u_3(t)e^{-t}$. Thus

$$g(t) = t^2 + u_3(t)(-t^2 + e^{-t})$$

Example 27. Rewrite $g(t) = |2t - 1|$ in step-function notation.

$$\begin{aligned} g(t) &= \begin{cases} -(2t-1) & \text{if } 2t - 1 < 0 \\ 2t - 1 & \text{if } 2t - 1 \geq 0 \end{cases} \\ &= \begin{cases} 1 - 2t & \text{if } t < \frac{1}{2} \\ 2t - 1 & \text{if } t \geq \frac{1}{2} \end{cases} \end{aligned}$$

The initial formula is $g(t) = 1 - 2t$. At time $t = \frac{1}{2}$, we use $u_{1/2}(t)$ to switch off $1 - 2t$ and to switch on $2t - 1$:

$$\begin{aligned} g(t) &= 1 - 2t + u_{1/2}(t)\left[-(1-2t) + (2t-1)\right] \\ &= 1 - 2t + u_{1/2}(t)[4t - 2] \end{aligned}$$

Example 28. Rewrite

$$g(t) = \begin{cases} 2 & \text{if } t < 1 \\ 3t & \text{if } 1 \leq t < 2 \\ 5 & \text{if } t \geq 2 \end{cases}$$

in step-function notation.

The initial formula is the constant 2 and there are two switching times: $t = 1$ and $t = 2$. At $t = 1$, we use $u_1(t)$ to switch off $g(t) = 2$ and switch on $g(t) = 3t$. At $t = 2$, we use $u_2(t)$ to switch off the previous formula $g(t) = 3t$ and switch on $g(t) = 5$. Thus

$$g(t) = 2 + u_1(t)(-2 + 3t) + u_2(t)(-3t + 5)$$

Step-function notation lets us rewrite any function defined in pieces as a sum of terms of the form $u_a(t)f(t)$ and reduces the calculation of the transform of such a function to finding the transform of this new kind of term.

Don't forget that $\mathcal{L}$ is an integral over the domain $t \geq 0$. Thus any formula that affects only negative values of t has no effect on the transform. Hence we only consider transforming terms of the form $u_a(t)f(t)$ with $a \geq 0$.

The definition of $\mathcal{L}$ gives

$$\begin{aligned}\mathcal{L}[u_a(t)f(t)] &= \int_0^\infty e^{-st}u_a(t)f(t)dt \\ &= \int_0^a e^{-st}(0)f(t)dt + \int_a^\infty e^{-st}(1)f(t)dt \\ &= \int_a^\infty e^{-st}f(t)dt\end{aligned}$$

This differs from the Laplace transform of $f(t)$ in that the lower limit of integration is not zero. However, if we set $\tau = t - a$, then $d\tau = dt$, $\tau = 0$ when $t = a$ and $\tau \to \infty$ as $t \to \infty$. Hence, by rewriting our integral in terms of τ, we have

$$\begin{aligned}\mathcal{L}[u_a(t)f(t)] &= \int_{\tau=0}^{\tau=\infty} e^{-s(\tau+a)}f(\tau+a)d\tau \\ &= e^{-as}\int_0^\infty e^{-s\tau}f(\tau+a)d\tau\end{aligned}$$

The last integral is equal to the transform of $f(t+a)$. We thus have the following:

$$\boxed{\text{Second shift formula (s-shifting): For } a \geq 0, \mathcal{L}[u_a(t)f(t)] = e^{-as}\mathcal{L}[f(t+a)]}$$

Example 29. Find $\mathcal{L}[u_2(t)e^{-t}]$.

Using s-shifting, we have

$$\begin{aligned}\mathcal{L}[u_2(t)e^{-t}] &= e^{-2s}\mathcal{L}[e^{-(t+2)}] \\ &= e^{-2s}e^{-2}\mathcal{L}[e^{-t}] && \text{(since the Laplace transform is linear)} \\ &= e^{-2(s+1)}\frac{1}{s+1}\end{aligned}$$

Example 30. Find $\mathcal{L}[|2t-1|]$.

We saw earlier that

$$|2t-1| = 1 - 2t + u_{1/2}(t)(4t-2)$$

Thus

$$\begin{aligned}\mathcal{L}[|2t-1|] &= \mathcal{L}[1] - 2\mathcal{L}[t] + e^{-s/2}\mathcal{L}\left[4\left(t+\frac{1}{2}\right)-2\right] \\ &= \frac{1}{s} - \frac{2}{s^2} + e^{-s/2}\mathcal{L}[4t] \\ &= \frac{1}{s} - \frac{2}{s^2} + \frac{4e^{-s/2}}{s^2}\end{aligned}$$

Exercise. Find $\mathcal{L}[g(t)]$, where

$$g(t) = \begin{cases} 2 & \text{if } t < 1 \\ 3t & \text{if } 1 \leq t < 2 \\ 5 & \text{if } t \geq 2 \end{cases}$$

If we set $f(t-a) = h(t)$, then the second shift formula says

$$\begin{aligned}\mathcal{L}[u_a(t)f(t-a)] &= \mathcal{L}[u_a(t)h(t)] \\ &= e^{-as}\mathcal{L}[h(t+a)] \\ &= e^{-as}\mathcal{L}[f(t)]\end{aligned}$$

We thus have the following:

$$\boxed{\text{Second shift formula (inverse version): If } \mathcal{L}^{-1}[F(s)] = f(t), \text{ then } \mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)}$$

Example 31. Find $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+4}\right]$.

We first find

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] \\ &= \frac{1}{2}\sin 2t \end{aligned}$$

Then the inverse transform version of the second shift formula says

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+4}\right] &= u_3(t)f(t-3) \\ &= u_3(t)\frac{1}{2}\sin(2(t-3)) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{2}\sin(2t-6) & \text{if } t \ge 3 \end{cases} \end{aligned}$$

Example 32. Solve the initial-value problem

$$x'' - x = \begin{cases} t & \text{if } t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$$

with initial conditions $x(0) = x'(0) = 0$.

We first rewrite the ODE using $u_1(t)$.

$$x'' - x = t + u_1(t)(-t)$$

Next we transform both sides and solve for $\mathcal{L}[x]$.

$$\begin{aligned} s^2\mathcal{L}[x] - sx(0) - x'(0) - \mathcal{L}[x] &= \frac{1}{s^2} + e^{-s}\mathcal{L}[-(t+1)] \\ (s^2-1)\mathcal{L}[x] &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) \\ \mathcal{L}[x] &= \frac{1}{s^2(s^2-1)} - e^{-s}\left(\frac{1}{s^2(s^2-1)} + \frac{1}{s(s^2-1)}\right) \end{aligned}$$

We need to use partial fractions.

$$\begin{aligned} \frac{1}{s^2(s+1)(s-1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1} \\ 1 &= As(s+1)(s-1) + B(s+1)(s-1) + Cs^2(s-1) + Ds^2(s+1) \end{aligned}$$

$s = 0$ $\quad 1 = B(-1)$ $\quad B = -1$

$s = 1$ $\quad 1 = D(2)$ $\quad D = \frac{1}{2}$

$s = -1$ $\quad 1 = C(-2)$ $\quad C = -\frac{1}{2}$

$s = 2$ $\quad 1 = A(2)(3)(1) + B(3)(1) + C(4)(1) + D(4)(3)$

$\quad 1 = 6A - 3 - 2 + 6$ $\quad A = 0$

$$\frac{1}{s^2(s+1)(s-1)} = -\frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1}$$

To find the second part, we use partial fractions again.

$$\begin{aligned}
\frac{1}{s(s+1)(s-1)} &= \frac{E}{s} + \frac{F}{s+1} + \frac{G}{s-1} \\
1 &= E(s+1)(s-1) + Fs(s-1) + Gs(s+1) \\
s = 0 \qquad 1 &= E(-1) \qquad E = -1 \\
s = 1 \qquad 1 &= G(2) \qquad G = \frac{1}{2} \\
s = -1 \qquad 1 &= F(-1)(-2) \qquad F = \frac{1}{2} \\
\frac{1}{s(s+1)(s-1)} &= -\frac{1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{s^2(s^2-1)} + \frac{1}{s(s^2-1)} &= -\frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1} - \frac{1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1} \\
&= -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}
\end{aligned}$$

We thus use the inverse transform to solve for x.
That is,

$$\begin{aligned}
x &= \mathcal{L}^{-1}\left[-\frac{1}{s^2} - \frac{1/2}{s+1} + \frac{1/2}{s-1}\right] - \mathcal{L}^{-1}\left[e^{-s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right)\right] \\
&= -t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - \mathcal{L}^{-1}\left[e^{-s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right)\right]
\end{aligned}$$

We need to calculate

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left[-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right] \\
&= -t - 1 + e^{t}
\end{aligned}$$

Using the inverse version of the second shift formula, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left[e^{-s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-1}\right)\right] &= u_1(t)f(t-1) \\
&= u_1(t)\left[-(t-1) - 1 + e^{t-1}\right] \\
&= u_1(t)\left[-t + e^{t-1}\right]
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
x &= -t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - u_1(t)\left[-t + e^{t-1}\right] \\
&= \begin{cases} -t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & \text{if } t < 1 \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - e^{t-1} & \text{if } t \geq 1 \end{cases}
\end{aligned}$$

Exercise. Check that both x and x' are continuous at 1.

6.4.1 Dirac delta function

The Dirac delta function $\delta(t)$ is characterised by the following two properties:

$$(1) \qquad \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

and

$$(2) \qquad \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t-a) = 0$ for $t \neq a$ and

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

Setting $f(t) = e^{-st}$, we have, for $a \geq 0$,

$$\int_{0}^{\infty} e^{-st}\delta(t-a)dt = \int_{-\infty}^{\infty} e^{-st}\delta(t-a)dt = e^{-as}$$

Thus, for $a \geq 0$,

$$\mathcal{L}[\delta(t-a)] = e^{-as}$$

Example 33. Solve the initial-value problem

$$x'' + 9x = 3\delta(t-\pi), \qquad x(0) = 1, x'(0) = 0$$

Taking the Laplace transform of the ODE, we have

$$\begin{aligned} s^2\mathcal{L}[x] - sx(0) - x'(0) + 9\mathcal{L}[x] &= 3e^{-\pi s} \\ (s^2+9)\mathcal{L}[x] - s &= 3e^{-\pi s} \\ \mathcal{L}[x] &= \frac{s}{s^2+9} + \frac{3e^{-\pi s}}{s^2+9} \end{aligned}$$

Using the translation property, we thus have

$$\begin{aligned} x &= \cos 3t + [\sin 3(t-\pi)]u_\pi(t) \\ &= \begin{cases} \cos 3t & t < \pi \\ \cos 3t + \sin 3(t-\pi) & t \geq \pi \end{cases} \end{aligned}$$

6.5 Convolution

Definition 6.3. *Given two functions $f(t)$ and $g(t)$, we define a new function, called the convolution of f and g, denoted $f * g$, by the rule*

$$(f*g)(t) = \int_0^t f(t-u)g(u)du$$

The formula assigns a numerical value, $(f * g)(t)$, to each specific value of t so that, as far as integration is concerned, t acts like a constant.

The limits of integration refer to u: we integrate from $u = 0$ to $u = t$.

In practice, we try to rewrite $f(t-u)$ in terms of functions of t and u alone before integrating.

Example 34. Find $(f * g)(t)$ when $f(t) = e^{2t}$ and $g(t) = e^{3t}$.

We have

$$\begin{aligned}(f * g)(t) &= \int_0^t f(t-u)g(u)du \\ &= \int_0^t e^{2(t-u)}e^{3u}du \\ &= e^{2t}\int_0^t e^{u}du \\ &= e^{2t}(e^t - 1) \\ &= e^{3t} - e^{2t}\end{aligned}$$

Convolution shares many properties with multiplication:

- the distributive law: $f * (c_1 g_1 + c_2 g_2) = c_1(f * g_1) + c_2(f * g_2)$
- the associative law: $f * (g * h) = (f * g) * h$
- the commutative law: $f * g = g * f$

However, not every property carries over. For example, the convolution of $g(t)$ with the constant $f(t) = 1$ is not $g(t)$:

$$(1 * g)(t) = \int_0^t g(u)du \neq g(t)$$

In Laplace transforms, convolution turns into products.

$\boxed{\text{If } f(t) \text{ and } g(t) \text{ both have Laplace transforms, then } \mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]}$

This also applies to inverse transforms.

$\boxed{\text{If } F(s) \text{ and } G(s) \text{ both have inverse Laplace transforms, then } \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[F(s)] * \mathcal{L}^{-1}[G(s)]}$

Example 35. Find $\mathcal{L}^{-1}\left[\frac{1}{(s-2)(s-3)}\right]$.

We could do this by partial fractions, of course. Alternatively, we can use convolution:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s-2)(s-3)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s-2)}\right] * \mathcal{L}^{-1}\left[\frac{1}{(s-3)}\right] \\ &= e^{2t} * e^{3t} \\ &= e^{3t} - e^{2t} && \text{(from the previous example)}\end{aligned}$$

Example 36. Find $\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$.

We can't use partial fractions here, since the function is already written in its partial fraction decomposition. However, we can think of this as the product of $\frac{s}{(s^2+1)}$ and $\frac{1}{(s^2+1)}$. Then

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] &= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right] \\
&= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\
&= \cos t * \sin t \\
&= \int_0^t \cos(t-u)\sin u du
\end{aligned}$$

We now use trigonometric identities to simplify this integral

$$\begin{aligned}
\int_0^t \cos(t-u)\sin u du &= \int_0^t (\cos t\cos u + \sin t \sin u)\sin u du \\
&= \cos t\int_0^t \cos u \sin u du + \sin t \int_0^t \sin^2 u du \\
&= \cos t\int_0^t \cos u \sin u du + \frac{1}{2}\sin t \int_0^t (1-\cos 2u) du
\end{aligned}$$

$$\begin{aligned}
w &= \sin u \\
\frac{dw}{du} &= \cos u \\
du &= \frac{dw}{\cos u}
\end{aligned}$$

Thus the integral is

$$\begin{aligned}
\int_0^t \cos(t-u)\sin u du &= \cos t\int_{u=0}^{u=t} w\cos u\frac{dw}{\cos u} + \frac{1}{2}\sin t\int_0^t (1-\cos 2u)du \\
&= \cos t\frac{w^2}{2}\bigg|_{u=0}^{u=t} + \frac{1}{2}\sin t\left[u - \frac{\sin 2u}{2}\right]_{u=0}^{u=t} \\
&= \cos t\frac{\sin^2 u}{2}\bigg|_{u=0}^{u=t} + \frac{1}{2}\sin t\left[u - \frac{\sin 2u}{2}\right]_{u=0}^{u=t} \\
&= \frac{1}{2}\cos t\sin^2 t + \frac{1}{2}\sin t\left[t - \frac{\sin 2t}{2}\right]
\end{aligned}$$

Exercise. Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$.

Example 37. Solve the initial-value problem

$$x'' + x = \cos t \qquad\qquad x(0) = x'(0) = 0$$

We transform both sides of the ODE:

$$\begin{aligned}
s^2\mathcal{L}[x] - sx(0) - x'(0) + \mathcal{L}[x] &= \mathcal{L}[\cos t] \\
(s^2+1)\mathcal{L}[x] &= \frac{s}{s^2+1} \\
\mathcal{L}[x] &= \frac{s}{(s^2+1)^2} \\
x &= \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] \\
&= \frac{1}{2}\cos t\sin^2 t + \frac{1}{2}\sin t\left[t - \frac{\sin 2t}{2}\right] \qquad \text{(from the previous example)}
\end{aligned}$$

Example 38. Solve the initial-value problem

$$x^{iv} - x = \begin{cases} 2 & \text{if } t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

with initial conditions $x(0) = x'(0) = x''(0) = 0$, $x'''(0) = 2$.

In step-function notation, the ODE reads

$$x^{iv} - x = 2 - 2u_3(t)$$

so its transform is

$$\begin{aligned} s^4\mathcal{L}[x] - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0) - \mathcal{L}[x] &= \frac{2}{s} - \frac{2}{s}e^{-3s} \\ (s^4 - 1)\mathcal{L}[x] &= 2 + \frac{2}{s} - \frac{2}{s}e^{-3s} \end{aligned}$$

Solving for $\mathcal{L}[x]$, we have

$$\mathcal{L}[x] = \frac{2}{s^4 - 1} + \frac{2}{s(s^4 - 1)} - \frac{2}{s(s^4 - 1)}e^{-3s}$$

Using partial fractions on the first term, we have

$$\begin{aligned} \frac{2}{s^4 - 1} &= \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs + D}{s^2 + 1} \\ 2 &= A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1) \\ s = 1 \qquad 2 &= A(2)(2) \qquad A = \frac{1}{2} \\ s = -1 \qquad 2 &= B(-2)(2) \qquad B = -\frac{1}{2} \\ s = 0 \qquad 2 &= A - B + D(-1) \\ 2 &= \frac{1}{2} + \frac{1}{2} - D \qquad D = -1 \\ s = 2 \qquad 2 &= A(3)(5) + B(5) + (2C + D)(3) \\ 2 &= \frac{15}{2} - \frac{5}{2} + 6C - 3 \qquad C = 0 \\ \frac{2}{s^4 - 1} &= \frac{1/2}{s-1} - \frac{1/2}{s+1} - \frac{1}{s^2+1} \end{aligned}$$

Thus the inverse transform is

$$\mathcal{L}^{-1}\left[\frac{2}{s^4 - 1}\right] = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - \sin t$$

The second term could also be decomposed into partial fractions. However, note that it is the same as the first term, multiplied by $\frac{1}{s}$. We can thus use convolution to obtain its inverse transform from the previous one.

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{2}{s(s^4 - 1)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s}\right] * \mathcal{L}^{-1}\left[\frac{2}{s^4 - 1}\right] \\ &= 1 * \left(\frac{1}{2}e^t - \frac{1}{2}e^{-t} - \sin t\right) \\ &= \int_0^t \left(\frac{1}{2}e^u - \frac{1}{2}e^{-u} - \sin u\right) du \\ &= \frac{1}{2}(e^t - 1) + \frac{1}{2}(e^{-t} - 1) + (\cos t - 1) \\ &= \frac{1}{2}e^t - \frac{1}{2}e^{-t} + \cos t - 2 \end{aligned}$$

Finally, the third term is an exponential times the second. We obtain is inverse transform from that of the second term by means of the second shift formula.

$$\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s(s^4-1)}\right] = u_3(t)\left(\frac{1}{2}e^{t-3} + \frac{1}{2}e^{-(t-3)} + \cos(t-3) - 2\right)$$

Combining all three terms, we have

$$x = e^t + \cos t - \sin t - 2 - u_3(t)\left(\frac{1}{2}e^{t-3} + \frac{1}{2}e^{-(t-3)} + \cos(t-3) - 2\right)$$

7 First-order differential equations

A way to solve an initial-value problem is to first attempt to solve the general problem and then apply the initial conditions to determine unknown constants. For a first-order problem, there will be one constant, for second-order problems there will be two, and so on.

7.1 First-order equations

7.1.1 Form

A general form of these equations is

$$\frac{dy}{dt} = f(y,t),$$

where we will assume that $f(y,t)$ is a reasonably well-behaved function of both y and t.

7.1.2 Conditions

Usually, we are also given a condition at some value of t, such as

$$y(0) = y_0$$

where y_0 is some constant.

We are looking for a function y whose derivative is the given function f. From calculus, we have

$$y = \int f(t)dt + c,$$

where c is an arbitrary constant. We will use the initial condition to determine c.

Example 39.

$$y' = \sin 2x \qquad\qquad y(\pi) = 0$$

Integrating, we have

$$y = -\frac{1}{2}\cos 2x + c,$$

which is the general solution. Applying the initial condition, we have

$$0 = -\frac{1}{2} + c,$$

so $c = \frac{1}{2}$. Hence the solution is

$$y = -\frac{1}{2}\cos 2x + \frac{1}{2}.$$

Notice that we solved a much larger problem than we needed to. The general solution gives us an infinite number of solutions, and then we pluck out just one.

7.1.3 Solutions

We do not have a procedure for solving the general first-order equation for arbitrary f. However, there are solutions for separable and linear equations which we consider next.

7.2 Separable equations

7.2.1 Form

If the equation

$$\frac{dy}{dt} = f(y,t)$$

can be expressed as a product of a function of y times a function of t, that is, in the form

$$\frac{dy}{dt} = Y(y)T(t)$$

we call the equation separable.

7.2.2 Solutions

The expression for a separable equation can be manipulated to give

$$\frac{dy}{dt} = Y(y)T(t)$$
$$\frac{dy}{Y(y)} = T(t)dt$$

and then integrated on both sides to give

$$\int \frac{dy}{Y(y)} = \int T(t)dt + K$$

where K is an arbitrary constant which will be evaluated by applying the condition at $t = 0$. Of course, the two integrations may be difficult or impossible.
The constant of integration K is absolutely vital here. Sometimes it does turn out to be zero, but that does not diminish its importance.

Example 40. Solve the population growth problem

$$\frac{dC}{dt} = kC.$$

Treating this as a separable equation, we rearrange to get

$$\frac{dC}{C} = kdt$$

Integrate both sides to get

$$\ln C = kt + K$$

Using the initial condition $C(0)$, we have

$$\ln C(0) = k \cdot 0 + K$$
$$\ln C = kt + \ln C(0)$$
$$C = C(0)e^{kt}$$

Example 41. Solve the equation $\frac{dy}{dx} = (1-2x)y^2$.

We use separation of variables:

$$\frac{dy}{y^2} = (1-2x)dx$$
$$\int y^{-2}dy = \int (1-2x)dx$$
$$-y^{-1} = x - x^2 + c$$
$$y = \frac{1}{x^2 - x - c}$$

Exercise. A more sophisticated model for population growth was

$$\frac{dC}{dt} = kC\left(1 - \frac{C}{C_{max}}\right)$$

Rearrange the equation to get

$$\frac{dC}{C - C^2/C_{max}} = kdt$$

Hint: use partial fractions.

7.3 Linear equations

7.3.1 Definition

If

$$\frac{dy}{dt} = f(y,t)$$

is linear in y (y appears by itself and not in a function) then the first-order equation is said to be linear.
We can't have y^2 or $\sqrt{y+1}$ or anything fancy.
However, we don't care about t, so

$$y' = t^3 y + \sqrt{t}$$

is linear (in y), whereas

$$y' = \frac{t}{y}$$

is not.

7.3.2 Form

A linear first-order equation can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

7.3.3 Integrating factors

We can use an integrating factor to make the equation integrable. Consider the function

$$F(t) = \exp\left(\int p(t)dt\right)$$

Note that

$$F'(t) = F(t)p(t)$$

We multiply both sides of the linear equation by $F(t)$.

$$F(t)\left(\frac{dy}{dt} + p(t)y\right) = F(t)q(t)$$

Consider now the expression

$$\frac{d}{dt}(F(t)y(t)) = F(t)\frac{dy}{dt} + \frac{dF}{dt}y(t) \qquad \text{product rule}$$

and so

$$\frac{d}{dy}(F(t)y(t)) = F(t)\frac{dy}{dt} + F(t)p(t)y(t)$$

Thus, the equation we want to solve has become

$$\frac{d}{dt}(F(t)y(t)) = F(t)q(t)$$

which can be integrated to give

$$\int d(F(t)y(t)) = \int F(t)q(t)dt$$

We can now solve to get

$$y(t) = \frac{1}{F(t)}\left(\int F(t)q(t)dt + K\right)$$

where K is a constant of integration.
The function $F(t)$ is called an integrating factor. This uses the product rule for derivatives, but in reverse. Again, the constant of integration is vital.

Example 42. Solve

$$y' + \frac{1}{1+x}y = 1+x, \ y(0) = 0.$$

The integrating factor is

$$F(x) = \exp\left(\int \frac{dx}{1+x}\right) = 1+x$$

Rewrite the equation as

$$(1+x)y' + y = (1+x)^2$$

or

$$\frac{d}{dx}((1+x)y) = (1+x)^2.$$

(See how the product rule "collapses" the left-hand side?)
Integrating both sides, we get

$$\int \frac{d}{dx}((1+x)y)dx = \int (1+x)^2 dx$$

$$(1+x)y = \frac{1}{3}(1+x)^3 + K$$

where K is an arbitrary constant. Now, solve for y to get

$$y(x) = \frac{1}{3}(1+x)^2 + \frac{K}{1+x}$$

Use the condition $y(0) = 0$ to get

$$K = -\frac{1}{3}$$

Thus the solution is

$$y(x) = \frac{1}{3}(1+x)^2 - \frac{1}{3}\cdot\frac{1}{1+x}$$

7.4 Exact solutions

We can generalise the idea of an integrating factor. What was really going on was that we used the integrating factor to put the equation into an exact form.

From calculus, if a function $u(x,y)$ has continuous partial derivatives, then its total or exact differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

In particular, if $u(x,y) = c$ (a constant), then $du = 0$.

Example 43. Suppose $u = x + x^2y^3 = c$. Find the differential equation for $\frac{dy}{dx}$.

We have

$$du = (1 + 2xy^3)dx + 3x^2y^2dy = 0.$$

Rearranging,

$$
\begin{aligned}
(1 + 2xy^3) &= -3x^2y^2\frac{dy}{dx} \\
\frac{dy}{dx} &= -\frac{1+2xy^3}{3x^2y^2}.
\end{aligned}
$$

What we've done here is "solved" a differential equation, only in reverse. This gives us a powerful solution method.

A first-order differential equation of the form

$$\boxed{M(x,y)dx + N(x,y)dy = 0} \tag{1}$$

is called <u>exact</u> if its left-hand side is the total or exact differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

of some function $f(x,y)$. Then the differential equation (1) can be written

$$du = 0.$$

By integration, we obtain the general solution of (1) in the form

$$\boxed{u(x,y) = c.} \tag{2}$$

Comparing (1) and (2), we see that (1) is exact if there is some function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M \qquad \text{and} \qquad \frac{\partial u}{\partial y} = N.$$

Suppose M and N are defined and have continuous first partial derivatives in a region of the xy-plane whose boundary is a closed curve having no self-intersections. Then

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial^2 u}{\partial y \partial x} \\
\frac{\partial N}{\partial x} &= \frac{\partial^2 u}{\partial x \partial y}.
\end{aligned}
$$

By continuity, the two second derivatives are equal. Thus

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

This condition is both necessary and sufficient for $Mdx + Ndy$ to be an exact differential.

Example 44. Show that

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0 \tag{3}$$

is exact.

We have

$$
\begin{aligned}
M &= x^3 + 3xy^2 & \qquad N &= 3x^2y + y^3 \\
\frac{\partial M}{\partial y} &= 6xy & \qquad \frac{\partial N}{\partial x} &= 6xy
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

If (1) is exact, the function $u(x,y)$ can be found either by guessing or by integrating twice, once with respect to each variable.

Integrating with respect to x, we have

$$u = \int M dx + k(y)$$

where y is regarded as a constant in this integration, so $k(y)$ plays the role of the arbitary constant.

Integrating with respect to y, we have

$$u = \int N dy + j(x)$$

where x is regarded as a constant in this integration, so $j(x)$ plays the role of the arbitary constant.

To determine $j(x)$, we derive $\frac{\partial u}{\partial x}$ to get $\frac{dj}{dx}$ and integrate.

Example 45. Find the solution to (3). Check that this solution solves the differential equation.

We have

$$\begin{aligned} u &= \int M dx + k(y) \\ &= \int (x^3 + 3xy^2) dx + k(y) \\ &= \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y). \end{aligned}$$

Differentiating this, we have

$$\frac{\partial u}{\partial y} = 3x^2y + \frac{dk}{dy} = N = 3x^2y + y^3.$$

Hence $\frac{dk}{dy} = y^3$. Thus $k = \frac{1}{4}y^4 + \bar{c}$. Thus

$$u(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = c.$$

To check, we can differentiate u implicitly with respect to x:

$$\begin{aligned} \frac{du}{dx} &= 0 \\ x^3 + 3xy^2 + 3x^2yy' + y^3y' &= 0 \\ x^3 + 3xy^2 + (3x^2y + y^3)\frac{dy}{dx} &= 0. \end{aligned}$$

Example 46. Solve

$$\cos x \sinh y \frac{dy}{dx} - \sin x \cosh y = 0, \qquad y(0) = 0$$

Check that the solution satisfies the differential equation.

We have

$$\begin{aligned} M &= -\sin x \cosh y \\ \frac{\partial M}{\partial y} &= -\sin x \sinh y \end{aligned} \qquad\qquad \begin{aligned} N &= \cos x \sinh y \\ \frac{\partial N}{\partial x} &= -\sin x \sinh y \end{aligned}$$

Thus the equation is exact.

To solve, we have

$$
\begin{aligned}
u &= -\int \sin x \cosh y dx + k(y) \\
&= \cos x \cosh y + k(y) \\
\frac{\partial u}{\partial y} &= \cos x \sinh y + \frac{dk}{dy} = N = \cos x \sinh y.
\end{aligned}
$$

Thus $\frac{dk}{dy} = 0$ so $k = \bar{c}$.

The general solution is then

$$u(x, y) = \cos x \cosh y = c.$$

The initial condition $y(0) = 0$ gives

$$\cos 0 \cosh 0 = 1 = c.$$

Thus the solution is $\cos x \cosh y = 1$.

Checking, we see that

$$
\begin{aligned}
(\cos x \cosh y)' &= -\sin x \cosh y + \cos x (\sinh y) y' = 0 \\
\cos 0 \cosh 0 &= 1
\end{aligned}
$$

(Don't forget to check the initial condition!)

What happens if the equation isn't exact? In this case, we *cannot* use this technique.

Example 47. Consider

$$y - xy' = 0$$

Show that the equation is not exact and that the integrating method fails.

We have

$$
\begin{aligned}
M &= y & N &= -x \\
\frac{\partial M}{\partial y} &= 1 & \frac{\partial N}{\partial x} &= -1.
\end{aligned}
$$

Thus the equation is not exact, since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Trying the integral method, we have

$$
\begin{aligned}
u &= \int M dx + k(y) = xy + k(y) \\
\frac{\partial u}{\partial y} &= x + k'(y).
\end{aligned}
$$

This should equal $N = -x$. But this is impossible, since $k(y)$ can only depend on y.

Exercise. Solve the last equation using a different method.

7.4.1 A special case of non-exact equations

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, eg $\xi(x)$, then an integrating factor is

$$\mu(x) = e^{\int \xi(x) dx}.$$

Equivalently, if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M}$ is a function of y only, eg $\phi(y)$, then an integrating factor is

$$\mu(y) = e^{\int \phi(y) dy}.$$

Example 48. Solve

$$(3xy - y^2)dx + (x^2 - xy)dy = 0.$$

First, let's see if the equation is exact. We have

$$\frac{\partial M}{\partial y} = 3x - 2y \qquad\qquad \frac{\partial N}{\partial x} = 2x - y$$

Since there are not equal, the equation is not exact.

However, we have

$$\begin{aligned} \xi &= \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \\ &= \frac{3x - 2y - 2x + y}{x(x-y)} \\ &= \frac{x-y}{x(x-y)} \\ &= \frac{1}{x} \end{aligned}$$

is solely a function of x. Hence we have the integrating factor

$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

Multiplying the original equation by the integrating factor gives us

$$(3x^2y - xy^2)dx + (x^3 - x^2y)dy = 0.$$

We then have

$$\frac{\partial \tilde{M}}{\partial y} = 3x^2 - 2xy \qquad\qquad \frac{\partial \tilde{N}}{\partial x} = 3x^2 - 2xy$$

Since these are equal, the revised equation is exact.

Solving, we have

$$\begin{aligned} u &= \int \tilde{M} dx + k(y) \\ &= x^3y - \frac{1}{2}x^2y^2 + k(y) \\ \frac{\partial u}{\partial y} &= x^3 - x^2y + k'(y) = \tilde{N} = x^3 - x^2y \\ \therefore \quad k'(y) &= 0 \\ k(y) &= K \\ u &= x^3y - \frac{1}{2}x^2y^2 + K = c \end{aligned}$$

so the solution (in implicit form) is

$$x^3y - \frac{1}{2}x^2y^2 = \tilde{c}$$

7.5 The Bernoulli Equation

Some nonlinear differential equations can be reduced to linear forms. The most famous of these is the Bernoulli equation:

$$\boxed{y' + p(x)y = g(x)y^a} \qquad (a \text{ any real number}). \tag{4}$$

If $a = 0$ or $a = 1$, the equation is linear. Otherwise it is nonlinear. In that case, set

$$u(x) = [y(x)]^{1-a}.$$

Differentiating and substituting into (4), we have

$$\begin{aligned} u' = (1-a)y^{-a}y' &= (1-a)y^{-a}(gy^a - py) \\ &= (1-a)(g - py^{1-a}). \end{aligned}$$

Since $u = y^{1-a}$, we thus have a linear equation

$$u' + (1-a)pu = (1-a)g.$$

Example 49. Solve

$$y' - Ay = -By^2 \qquad (A, B \text{ positive constants}).$$

Here $a = 2$, so $u = y^{-1}$ and we thus have

$$\begin{aligned} u' = -y^{-2}y' &= -y^{-2}(-By^2 + Ay) \\ &= B - Ay^{-1} \\ &= B - Au \\ u' + Au &= B \end{aligned}$$

This is a linear equation, so we can use an integrating factor e^{Ax}:

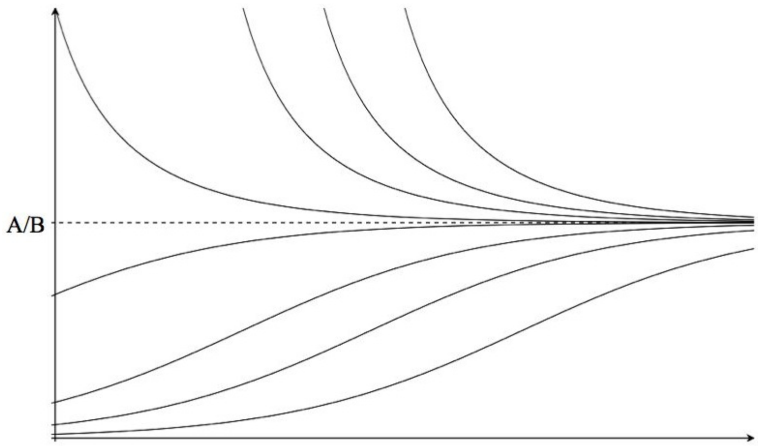
$$\begin{aligned} e^{Ax}u' + Ae^{Ax}u &= Be^{Ax} \\ \frac{d}{dx}(e^{Ax}u) &= Be^{Ax} \\ \int \frac{d}{dx}(e^{Ax}u) &= \int Be^{Ax} \\ e^{Ax}u &= \frac{B}{A}e^{Ax} + c \\ u &= \frac{B}{A} + ce^{-Ax}. \end{aligned}$$

We thus have

$$y = \frac{1}{u} = \frac{1}{\frac{B}{A} + ce^{-Ax}}.$$

Another (trivial) solution is $y = 0$. The nontrivial solution is called the logistic law of population growth, where x represents time. For $B = 0$, it gives exponential growth $y = (1/c)e^{Ax}$, which is Malthus's law.

Exponential growth is of course not very realistic, as solutions head to infinity very quickly. Thus, the term $-By^2$ in the original differential equation acts as a "braking term", preventing the population from growing without bound. Indeed, small populations (with $0 < y(0) < A/B$) will increase monotonically to A/B, whereas initial conditions with $y(0) > A/B$ will decrease monotonically to the same limit A/B. The value A/B is called the carrying capacity of a population.


A/B